Use Laplace transforms to solve the IVP $y'' + 4y' + 13y = 39, \quad y(0) = -4, \quad y'(0) = 2$. **SCORE: _____ / 40 PTS**

3 EACH EXCEPT AS OTHERWISE NOTED

$$s^2Y - sy(0) - y'(0) + 4sY - 4y(0) + 13Y = (s^2+4s+13)Y + 4s + 14 = \frac{39}{s}$$

2

$$Y = \frac{-4s-14}{(s+2)^2+3^2} + \frac{39}{s((s+2)^2+3^2)}$$

$$\frac{-4s-14}{(s+2)^2+3^2} = \frac{-4(s+2)-2(3)}{(s+2)^2+3^2}$$

6

$$\frac{39}{s((s+2)^2+3^2)} = \frac{A}{s} + \frac{B(s+2)+C(3)}{(s+2)^2+3^2} \longrightarrow \frac{3}{s} + \frac{-3(s+2)-2(3)}{(s+2)^2+3^2}$$

$$39 = A[(s+2)^2+3^2] + B(s+2)s + C(3s)$$

$s=0$: $39 = 13A \rightarrow A=3$

$s=-2$: $39 = 9A - 6C \rightarrow 39 = 27 - 6C \rightarrow C=-2$

s^2: $0 = A + B \rightarrow B = -3$

SANITY CHECK $s=2$: $\frac{39}{2(25)} \stackrel{?}{=} \frac{3}{2} + \frac{-3(4)-2(3)}{25}$

$$\frac{39}{50} = \frac{3}{2} - \frac{18}{25} = \frac{75-36}{50} \checkmark$$

$$y = -4e^{-2t}\cos 3t - 2e^{-2t}\sin 3t + 3 - 3e^{-2t}\cos 3t - 2e^{-2t}\sin 3t$$

$$= -7e^{-2t}\cos 3t - 4e^{-2t}\sin 3t + 3$$

2

Find two linearly independent series solutions of $9x^2y''+9x^2y'+2y=0$ about the point $x=0$. **SCORE: _____ / 50 PTS**

ORDINARY SINGULAR POINT

You must find the recurrence relation(s) for the coefficients
and you must give the first four non-zero terms of each series (or all terms if a solution has fewer than four non-zero terms), but you do NOT need to write your final answers in sigma notation.

$$9x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}+9x^2\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}+2\sum_{n=0}^{\infty}a_nx^{n+r}=0$$

$$\sum_{n=0}^{\infty}9(n+r)(n+r-1)a_nx^{n+r}+\sum_{n=0}^{\infty}9(n+r)a_nx^{n+r+1}+\sum_{n=0}^{\infty}2a_nx^{n+r}=0$$

$$\sum_{n=1}^{\infty}9(n+r-1)a_{n-1}x^{n+r}$$

2 EACH EXCEPT AS OTHERWISE NOTED

$$9r(r-1)a_0x^r+2a_0x^r+\sum_{n=1}^{\infty}[(9(n+r)(n+r-1)+2)a_n+9(n+r-1)a_{n-1}]x^{n+r}=0$$

$$9r(r-1)+2=0$$
$$9r^2-9r+2=0$$
$$(3r-1)(3r-2)=0$$
$$r=\tfrac{1}{3},\tfrac{2}{3}$$

$$a_n=-\frac{9(n+r-1)}{9(n+r)(n+r-1)+2}a_{n-1},\quad n\geq 1$$

4

$$r=\tfrac{1}{3}:\ a_n=-\frac{9(n-\frac{2}{3})}{9(n+\frac{1}{3})(n-\frac{2}{3})+2}a_{n-1}=-\frac{3(3n-2)}{(3n+1)(3n-2)+2}a_{n-1}$$

$$=-\frac{3(3n-2)}{9n^2-3n}a_{n-1}=-\frac{3n-2}{n(3n-1)}a_{n-1}$$ 4

$$a_0=1\rightarrow a_1=-\frac{1}{1\cdot 2}\rightarrow a_2=\frac{4\cdot 1}{2\cdot 1\cdot 5\cdot 2}\rightarrow a_3=-\frac{7\cdot 4\cdot 1}{3\cdot 2\cdot 1\cdot 8\cdot 5\cdot 2}$$

$$y_1=x^{\frac{1}{3}}\left(1-\frac{1}{1\cdot 2}x+\frac{4\cdot 1}{2\cdot 1\cdot 5\cdot 2}x^2-\frac{7\cdot 4\cdot 1}{3\cdot 2\cdot 1\cdot 8\cdot 5\cdot 2}x^3+\dots\right)$$

$$r=\tfrac{2}{3}:\ a_n=-\frac{9(n-\frac{1}{3})}{9(n+\frac{2}{3})(n-\frac{1}{3})+2}a_{n-1}=-\frac{3(3n-1)}{(3n+2)(3n-1)+2}a_{n-1}$$

$$=-\frac{3(3n-1)}{9n^2+3n}a_{n-1}=-\frac{3n-1}{n(3n+1)}a_{n-1}$$ 4

$$a_0=1\rightarrow a_1=-\frac{2}{1\cdot 4}\rightarrow a_2=\frac{5\cdot 2}{2\cdot 1\cdot 7\cdot 4}\rightarrow a_3=-\frac{8\cdot 5\cdot 2}{3\cdot 2\cdot 1\cdot 10\cdot 7\cdot 4}$$

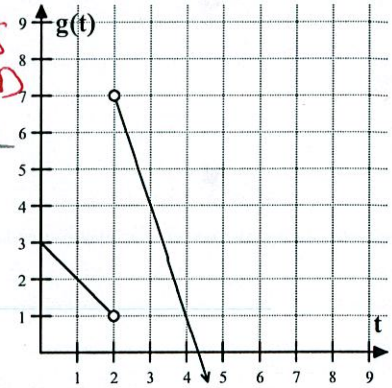
$$y_2=x^{\frac{2}{3}}\left(1-\frac{2}{1\cdot 4}x+\frac{5\cdot 2}{2\cdot 1\cdot 7\cdot 4}x^2-\frac{8\cdot 5\cdot 2}{3\cdot 2\cdot 1\cdot 10\cdot 7\cdot 4}x^3+\dots\right)$$

Use Laplace transforms to solve the IVP $y'' - y = g(t)$, $y(0) = -1$, $y'(0) = -3$, **SCORE: _____ / 60 PTS**

where $g(t)$ is the function whose graph is shown on the right.

2½ EACH EXCEPT AS OTHERWISE NOTED

NOTE: Each "piece" of $g(t)$ is linear.



$$g(t) = \begin{cases} 3-t, & 0<t<2 \\ 13-3t, & t>2 \end{cases} = 3-t+\begin{cases} 0, & 0<t<2 \\ 10-2t, & t>2 \end{cases}$$

$$= 3-t+u(t-2)(10-2t)$$

$$\mathcal{L}\{g(t)\} = \frac{3}{s} - \frac{1}{s^2} + e^{-2s}\mathcal{L}\{10-2(t+2)\}$$

$$= \frac{3}{s} - \frac{1}{s^2} + e^{-2s}\mathcal{L}\{6-2t\}$$

$$= \frac{3}{s} - \frac{1}{s^2} + e^{-2s}\left(\frac{6}{s} - \frac{2}{s^2}\right) = \frac{3s-1}{s^2} + 2e^{-2s}\frac{3s-1}{s^2}$$

$$s^2Y - sy(0) - y'(0) - Y = (s^2-1)Y + s + 3 = \frac{3s-1}{s^2} + 2e^{-2s}\frac{3s-1}{s^2}$$

$$Y = \frac{-s-3}{(s+1)(s-1)} + \frac{3s-1}{s^2(s+1)(s-1)} + 2e^{-2s}\frac{3s-1}{s^2(s+1)(s-1)}$$

10

$$\frac{-s-3}{(s+1)(s-1)} = \frac{A}{s+1} + \frac{B}{s-1}$$

$$-s-3 = A(s-1) + B(s+1)$$

$s=1$: $-4 = 2B \rightarrow B=-2$

$s=-1$: $-2 = -2A \rightarrow A=1$

CHECK $s=2$: $\frac{-5}{3} \stackrel{?}{=} \frac{1}{3} + \frac{-2}{1}$ ✓

$$\frac{1}{s+1} - \frac{2}{s-1}$$ 5

$$\frac{3s-1}{s^2(s+1)(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s-1}$$

$$\frac{-3}{s} + \frac{1}{s^2} + \frac{2}{s+1} + \frac{1}{s-1}$$

$$3s-1 = As(s+1)(s-1) + B(s+1)(s-1) + Cs^2(s-1) + Ds^2(s+1)$$

$s=0$: $-1 = -B \rightarrow B=1$

$s=1$: $2 = 2D \rightarrow D=1$

$s=-1$: $-4 = -2C \rightarrow C=2$

s^3: $0 = A+C+D \rightarrow A = -(C+D) = -3$

CHECK $s=2$: $\frac{5}{4(3)} \stackrel{?}{=} \frac{-3}{2} + \frac{1}{4} + \frac{2}{3} + \frac{1}{1}$

$$\frac{5}{12} \stackrel{?}{=} \frac{-18+3+8+12}{12}$$ ✓

5 5

$$y = e^{-t} - 2e^{t} - 3 + t + 2e^{-t} + e^{t} + 2u(t-2)\left(-3+t-2+2e^{-(t-2)} + e^{t-2}\right)$$

$$= 3e^{-t} - e^{t} - 3 + t + 2u(t-2)\left(-5+t+2e^{-(t-2)} + e^{t-2}\right)$$